

# DIFFUSION PROCESSES ON BRANCHING BROWNIAN MOTION

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**ABSTRACT.** We construct a class of one-dimensional diffusion processes on the particles of branching Brownian motion that are symmetric with respect to the limits of random martingale measures. These measures are associated with the extended extremal process of branching Brownian motion and are supported on a Cantor-like set. The processes are obtained via a time-change of a standard one-dimensional reflected Brownian motion on  $\mathbb{R}_+$  in terms of the associated positive continuous additive functionals.

The processes introduced in this paper may be regarded as an analogue of the Liouville Brownian motion which has been recently constructed in the context of a Gaussian free field.

## 1. INTRODUCTION

Over the last years diffusion processes in random environment, constructed by a random time-change of a standard Brownian motion in terms of singular measures, appeared in several situations. One prime example is the so-called FIN-diffusion (for Fontes, Isopi and Newman) introduced in [22] which appears for instance as the annealed scaling limit for one-dimensional trap models (see [22, 6, 7]) and for the one-dimensional random conductance model with heavy-tailed conductances (see [36, Appendix A]). Another example is the Liouville Brownian motion, recently constructed in [25, 8] as the natural diffusion process in the random geometry associated with two-dimensional Liouville quantum gravity.

In this paper we add one more class of examples to the collection. We explicitly construct the time change as the right-continuous inverse of the positive continuous additive functional whose Revuz measure is the limit of certain random martingale measures that appear in the description of the extremal process of a branching Brownian motion (BBM for short). As a result we obtain a pure jump diffusion process on a Cantor-like set representing the positions of the BBM particles in the underlying Galton-Watson tree.

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*Date:* April 25, 2017.

*2010 Mathematics Subject Classification.* Primary: 60J55, 60J80, 60K37; Secondary: 60G55, 60G70, 60J60.

*Key words and phrases.* Branching Brownian motion, additive functional, extremal process, local time, random environment.

This work has been written while the authors were affiliated to the Rheinische Friedrich-Wilhelms Universität Bonn and were partially supported by the German Research Foundation in the Collaborative Research Center 1060 “The Mathematics of Emergent Effects”, Bonn.

Branching Brownian motion has already been introduced in [30, 34] in the late 1950s and early 1960s. It is a continuous-time Markov branching process on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  which is constructed as follows. We start with a continuous-time Galton-Watson process (see e.g. [5]) with branching mechanism  $p_k, k \geq 1$ , normalised such that  $\sum_{k=1}^{\infty} p_k = 1$ ,  $\sum_{k=1}^{\infty} k p_k = 2$  and  $K = \sum_{k=1}^{\infty} k(k-1)p_k < \infty$ . At any time  $t$  we may label the endpoints of the process  $i_1(t), \dots, i_{n(t)}(t)$ , where  $n(t)$  is the number of branches at time  $t$ . Observe that by our choice of normalisation we have that  $\mathbb{E}n(t) = e^t$ . BBM is then constructed by starting a Brownian motion at the origin at time zero, running it until the first time the GW process branches, and then starting independent Brownian motions for each branch of the GW process starting at the position of the original BM at the branching time. Each of these runs again until the next branching time of the GW occurs, and so on.

We denote the positions of the  $n(t)$  particles at time  $t$  by  $x_1(t), \dots, x_{n(t)}(t)$ . Note that, of course, the positions of these particles do not reflect the position of the particles "in the tree".

*Remark 1.1.* By a slight abuse of notation, we also denote by  $x_k(s)$  for  $s < t$  the particle position of the ancestor of the particle  $i_k(t)$  at time  $s$ .

Setting  $m(t) := \sqrt{2}t - \frac{3}{2\sqrt{2}} \log(t)$ , Bramson [15, 14], and Lalley and Selke [28] showed that

$$\lim_{t \uparrow \infty} \mathbb{P} \left( \max_{k \leq n(t)} x_k(t) - m(t) \leq x \right) = \mathbb{E} \left[ e^{-CZ e^{-\sqrt{2}x}} \right], \quad (1.1)$$

for some constant  $C$ , where  $Z := \lim_{t \uparrow \infty} Z_t$  is the  $\mathbb{P}$ -a.s. limit of the derivative martingale

$$Z_t := \sum_{j \leq n(t)} (\sqrt{2}t - x_j(t)) e^{\sqrt{2}(x_j(t) - \sqrt{2}t)}, \quad t \geq 0. \quad (1.2)$$

For  $0 < r < t$  a truncated version of the derivative martingale

$$Z(v, r, t) := \sum_{j \leq n(t)} (\sqrt{2}t - x_j(t)) e^{\sqrt{2}(x_j(t) - \sqrt{2}t)} \mathbb{1}_{\{\gamma(x_j(r)) \leq v\}}, \quad v \in \mathbb{R}_+, \quad (1.3)$$

has been introduced in [12]. Here we denote by  $\gamma$  an embedding of the particles  $\{1, \dots, n(t)\}$  into  $\mathbb{R}_+$ , which encodes the positions of the particles in the underlying Galton-Watson tree respecting the genealogical distance (see Section 2.1 below for the precise definition). The associated random measure on  $\mathbb{R}_+$  is given by

$$M_{r,t} := \sum_{j \leq n(t)} (\sqrt{2}t - x_j(t)) e^{\sqrt{2}(x_j(t) - \sqrt{2}t)} \delta_{\gamma(x_j(r))}. \quad (1.4)$$

In [12] it has been shown that the weak limit

$$M = \lim_{r \uparrow \infty} \lim_{t \uparrow \infty} M_{r,t} \quad \text{exists } \mathbb{P}\text{-a.s.} \quad (1.5)$$

Furthermore, in [12, Theorem 3.1] an extended convergence result of the extremal process has been proven, namely

$$\sum_{k=1}^{n(t)} \delta_{(\gamma(x_k(t), x_k(t) - m(t)))} \Rightarrow \sum_{i,j} \delta_{(q_i, p_i) + (0, \Delta_j^{(i)})}, \quad \text{on } \mathbb{R}_+ \times \mathbb{R}, \text{ as } t \uparrow \infty, \quad (1.6)$$

where  $(q_i, p_i)_{i \in \mathbb{N}}$  are the atoms of a Cox process on  $\mathbb{R}_+ \times \mathbb{R}$  with intensity measure  $M(dv) \times C e^{-\sqrt{2}x} dx$  and  $(\Delta_j^{(i)})_{i,j}$  are the atoms of independent and identically distributed point processes  $\Delta^{(i)}$  with

$$\Delta^{(1)} \stackrel{D}{=} \lim_{t \uparrow \infty} \sum_{i=1}^{n(t)} \delta_{\tilde{x}_i(t) - \max_{j \leq n(t)} \tilde{x}_j(t)}, \quad (1.7)$$

where  $\tilde{x}(t)$  is a BBM conditioned on  $\max_{j \leq n(t)} \tilde{x}_j(t) \geq \sqrt{2}t$ . Recall that in [4, 1] it was already shown that  $\sum_{k=1}^{n(t)} \delta_{x_k(t) - m(t)}$  converges to the Poisson cluster process given by the projection of the limit in (1.6) onto the second coordinate.

**1.1. Results.** Let  $(\Omega', (B_s)_{s \geq 0}, \mathcal{G}, (\mathcal{G}_s)_{s \geq 0}, (P_x)_{x \in \mathbb{R}_+})$  denote a one-dimensional reflected standard Brownian motion  $B$  on  $\mathbb{R}_+$ . Recall that  $B$  is reversible w.r.t. the Lebesgue measure  $dx$  on  $\mathbb{R}_+$ . Then, the positive continuous additive functional (PCAF) of  $B$  having Revuz measure  $M_{r,t}$  (see Appendix A for definitions) is given by  $F_{r,t} : [0, \infty) \rightarrow [0, \infty)$  defined as

$$F_{r,t}(s) := \int_{\mathbb{R}_+} L_s^a M_{r,t}(da) = \sum_{j=1}^{n(t)} (\sqrt{2}t - x_j(t)) e^{\sqrt{2}(x_j(t) - \sqrt{2}t)} L_s^{\gamma(x_j(t))}, \quad (1.8)$$

where  $\{L^a, a \in \mathbb{R}\}$  denotes the family of local times of  $B$ . Now we state the main result of the paper.

**Theorem 1.2.** *For any  $S > 0$  and  $x \in \mathbb{R}_+$  the following hold.*

- (i) *The limit  $F = \lim_{r \uparrow \infty} \lim_{t \uparrow \infty} F_{r,t}$  exists in sup-norm on  $[0, S]$  in  $\mathbb{P} \times P_x$ -probability.*
- (ii) *The limiting functional  $F$  is  $\mathbb{P} \times P_x$ -a.s. explicitly given by*

$$F(s) = \int_{\mathbb{R}_+} L_s^a M(da), \quad s \geq 0, \quad (1.9)$$

*and  $F$  is  $\mathbb{P}$ -a.s. the (up to equivalence) unique PCAF with Revuz measure  $M$ . In particular,  $F$  is  $\mathbb{P} \times P_x$ -a.s. continuous, increasing and satisfies  $F(0) = 0$  and  $\lim_{s \rightarrow \infty} F(s) = \infty$ .*

**Remark 1.3.** Theorem 1.2 cannot be directly derived from [35, Theorem 1 (3)]. The speed measures considered in [35] are deterministic, while in our situation the measures  $M_{r,t}$  are random with *random* support. Hence, it is not clear how to verify condition (iv) in [35]  $\mathbb{P}$ -a.s. simultaneously for all sequences of points in the supports of the measures  $M_{r,t}$ .

**Definition 1.4.** We define the process  $\mathcal{B}$  as the time-changed Brownian motion

$$\mathcal{B}(s) := B_{F^{-1}(s)}, \quad s \geq 0, \quad (1.10)$$

where  $F^{-1}$  denotes the right-continuous inverse of the PCAF  $F$  in Theorem 1.2.

By the general theory of time changes of Markov processes, in particular cf. [24, Theorem 6.2.1],  $\mathcal{B}$  is a right-continuous strong Markov process on  $\text{supp } M$ , which is  $M$ -symmetric and induces a strongly continuous transition semigroup. Note that the empty set is the only polar set for the one-dimensional Brownian motion, so the measure  $M$  does trivially not charge polar sets. Further, for any  $0 < r < t$  set

$$\mathcal{B}_{r,t}(s) := B_{F_{r,t}^{-1}(s)}, \quad s \geq 0, \quad (1.11)$$

where  $F_{r,t}^{-1}$  denotes the right-continuous inverse of  $F_{r,t}$ . Then, as  $r$  and  $t$  tend to infinity, under the annealed measure the processes  $\mathcal{B}_{r,t}$  converge in law towards  $\mathcal{B}$  on the Skorohod space  $D([0, \infty), \mathbb{R}_+)$  equipped with  $L_{\text{loc}}^1$ -topology (see Theorem 4.1 below).

In order to prove Theorem 1.2 we first take the limit of  $F_{r,t}$  for  $t \uparrow \infty$  using the branching property together with certain martingale limits. To show the existence of the limit in  $r$ , we exploit the appearance of the measure  $M$  in the description of the extended extremal process. The argument is based on a thinning technique combined with certain properties of the embedding  $\gamma$  and the Hölder continuity of Brownian local times. Finally, the limiting functional is identified as the PCAF having Revuz measure  $M$ , which is unique by general theory.

Similarly to the above procedure, for any  $\sigma \in (0, 1)$  one obtains a measure  $M^\sigma$  from a truncation of the McKean martingale

$$Y_t^\sigma := \sum_{i=1}^{n(t)} e^{\sqrt{2}\sigma x_k(t) - (1+\sigma^2)t}, \quad t \geq 0. \quad (1.12)$$

Then one can construct the PCAF  $F^\sigma$  associated with  $M^\sigma$  and define the process  $\mathcal{B}^\sigma$  as  $\mathcal{B}^\sigma(s) := B_{(F^\sigma)^{-1}(s)}$ . We refer to Section 5 for further details.

A diffusion process being similar to but different from  $\mathcal{B}$  is the FIN-diffusion introduced in [22]. It is a one-dimensional singular diffusion in random environment given by a random speed measure  $\rho = \sum_i v_i \delta_{y_i}$ , where  $(y_i, v_i)$  is an inhomogeneous Poisson point process on  $\mathbb{R} \times (0, \infty)$  with intensity measure  $dy \alpha v^{-1-\alpha} dv$  for  $\alpha \in (0, 1)$ . Let  $F_{\text{FIN}}$  be the PCAF

$$F_{\text{FIN}}(s) := \int_{\mathbb{R}} L_s^a(W) \rho(da) \quad (1.13)$$

with  $\{L^a(W), a \in \mathbb{R}\}$  denoting the family of local times of a one-dimensional Brownian motion  $W$ . Then, the FIN-diffusion  $\{\text{FIN}(s), s \geq 0\}$  is the diffusion process defined as the time change  $\text{FIN}(s) := W_{(F_{\text{FIN}})^{-1}(s)}$  of the Brownian motion  $W$ . At first sight the measure  $\rho$  and the process FIN resemble strongly  $M$  and  $\mathcal{B}$ , respectively. However, one significant difference is that  $\rho$  is a discrete random measure

with a set of atoms being dense in  $\mathbb{R}$ , so that  $\rho$  has full support  $\mathbb{R}$  and FIN has continuous sample paths (see [22] or [7, Proposition 3.2]), while the measure  $M$  is concentrated on a Cantor-like set and the sample paths of  $\mathcal{B}$  have jumps.

Another prominent example for a log-correlated process is the Gaussian Free Field (GFF) on a two-dimensional domain. In a sense the processes  $\mathcal{B}$  or  $\mathcal{B}^\sigma$  introduced in this paper can be regarded as the BBM-analogue of the Liouville Brownian motion (LBM) recently constructed in [25] and in a weaker form in [8]. More precisely, let  $X$  denote a (massive) GFF on a domain  $D \subseteq \mathbb{R}^2$ , then in the subcritical case the analogue of the martingale measure  $M^\sigma$  can be constructed by using the theory of Gaussian multiplicative chaos established by Kahane in [27] (see also [32] for a review). On a formal level the resulting so-called Liouville measure on  $D$  is given by

$$e^{\gamma X(z) - \frac{\gamma^2}{2} \mathbb{E}[X(z)^2]} dz, \quad \gamma \in (0, 2). \quad (1.14)$$

The associated PCAF  $F_{\text{LBM}}$ , which can formally be written as

$$F_{\text{LBM}}(s) = \int_0^s e^{\gamma X(W_r) - \frac{\gamma^2}{2} \mathbb{E}[X(W_r)^2]} dr, \quad (1.15)$$

where  $W$  denotes a two-dimensional standard Brownian motion on the domain  $D$ , has been constructed in [25] (cf. also [2, Appendix A]). Then, the Liouville Brownian motion  $\{\text{LBM}(s), s \geq 0\}$  is defined as  $\text{LBM}(s) := W_{F_{\text{LBM}}^{-1}(s)}$ .

In the critical case  $\gamma = 2$  the corresponding analogue of the derivative martingale measure  $M$  can be interpreted as being given by

$$-2(X(z) - \mathbb{E}[X(z)^2]) e^{2(X(z) - \mathbb{E}[X(z)^2])} dz, \quad (1.16)$$

which has been introduced in [20, 21]. The corresponding PCAF and the critical Liouville Brownian motion have been constructed in [33]. In the context of a discrete GFF such measures have been studied in [10, 9, 11], where in [10] an analogue of the extended convergence result in (1.6) has been established.

However, a major difference between the processes  $\mathcal{B}$  and LBM is that for the LBM the functional  $F_{\text{LBM}}$  and the planar Brownian motion  $W$  are independent (cf. [25, Theorem 2.21]), while in the present paper the functional  $F$  and the Brownian motion  $B$  are *dependent* since  $L$  is the local time of  $B$ . A similar phenomenon can be observed in the context of trap models, where in dimension  $d = 1$  the underlying Brownian motion and the clock process of the FIN diffusion are dependent and in dimension  $d \geq 2$  the Brownian motion and the clock process of the scaling limit, known as the so-called fractional kinetics motion, are independent.

In [17] Croydon, Hambly and Kumagai consider time-changes of stochastic processes and their discrete approximations in a quite general framework for the case

when the underlying process is strongly recurrent, meaning that it can be described in terms of its resistance form (examples include the one-dimensional standard Brownian motion or Brownian motion on tree-like spaces and certain low-dimensional fractals). The results cover the FIN-diffusion and a one-dimensional version of the LBM. However, the results of the present paper do not immediately follow from the approximation result in [17] since the required convergence of the measures  $M_{r,t}$  towards  $M$  in the Gromov-Hausdorff-vague topology on the non-compact space  $\mathbb{R}_+$  is not obvious.

The rest of the paper is organised as follows. In Section 2 we provide the precise definition of the embedding  $\gamma$ , the (truncated) critical martingale measures and the associated PCAFs. Then we prove Theorem 1.2 in Section 3 and we specify some properties of the process  $\mathcal{B}$  in Section 4, in particular we describe its Dirichlet form. In Section 5 we sketch the construction of the process  $\mathcal{B}^\sigma$  associated with the martingale measure obtained from the McKean martingale. Finally, in the appendix we recall the definitions of a PCAF and its Revuz measure and collect some properties of Brownian local times needed in the proofs.

## 2. PRELIMINARIES

**2.1. Definition of the embedding.** We start by recalling the definition of the embedding  $\gamma$  given in [12] which is a slight variant of the familiar Ulam-Neveu-Harris labelling (see e.g. [26]). We denote the set of (infinite) multi-indices by  $\mathbf{I} \equiv \mathbb{Z}_+^\mathbb{N}$ , and let  $\mathbf{F} \subset \mathbf{I}$  be the subset of multi-indices that contain only finitely many entries different from zero. Ignoring leading zeros, we see that

$$\mathbf{F} = \cup_{k=0}^{\infty} \mathbb{Z}_+^k, \quad (2.1)$$

where  $\mathbb{Z}_+^0$  is either the empty multi-index or the multi-index containing only zeros.

We encode a continuous-time Galton-Watson process by the set of branching times,  $\{t_1 < t_2 < \dots < t_{W(t)} < \dots\}$ , where  $W(t)$  denotes the number of branching times up to time  $t$ , and by a consistently assigned set of multi-indices for all times  $t \geq 0$ . To do so, (for a given tree) the sets of multi-indices,  $\tau(t)$  at time  $t$ , are constructed as follows.

- $\{(0, 0, \dots)\} = \{u(0)\} = \tau(0)$ .
- for all  $j \geq 0$ , for all  $t \in [t_j, t_{j+1})$ ,  $\tau(t) = \tau(t_j)$ .
- If  $u \in \tau(t_j)$  then  $u + \underbrace{(0, \dots, 0, k, 0, \dots)}_{W(t_j) \times 0} \in \tau(t_{j+1})$  if  $0 \leq k \leq l^u(t_{j+1}) - 1$ ,

where

$$l^u(t_j) = \#\{\text{offsprings of the particle corresponding to } u \text{ at time } t_j\}. \quad (2.2)$$

We use the convention that, if a given branch of the tree does not "branch" at time  $t_j$ , we add to the underlying Galton-Watson at this time an extra vertex where  $l^u(t_j) = 1$  (see Figure 1). We call the resulting tree  $\tilde{T}_t$ .

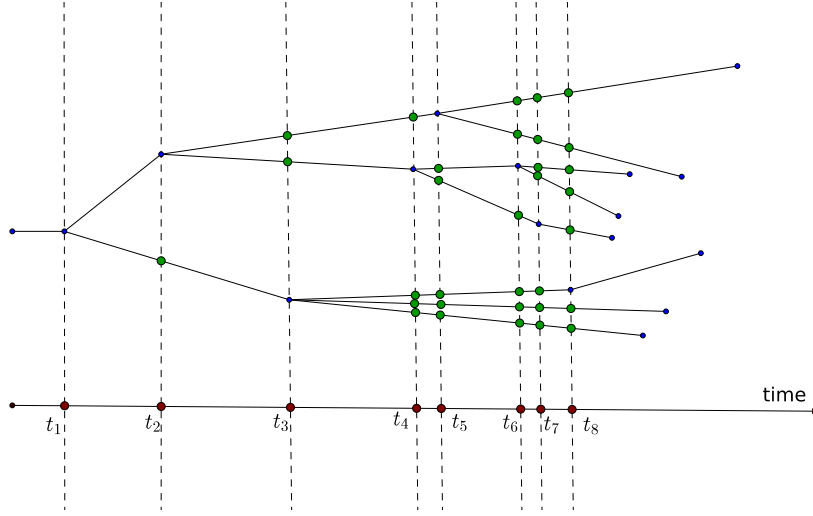


FIGURE 1. Construction of  $\tilde{T}$ : The green nodes were introduced into the tree ‘by hand’.

One relates the assignment of labels in the following backward consistent way. For  $u \equiv (u_1, u_2, u_3, \dots) \in \mathbb{Z}_+^{\mathbb{N}}$ , we define the function  $u(r)$ ,  $r \in \mathbb{R}_+$ , through

$$u_\ell(r) \equiv \begin{cases} u_\ell, & \text{if } t_\ell \leq r, \\ 0, & \text{if } t_\ell > r. \end{cases} \quad (2.3)$$

Clearly, if  $u(t) \in \tau(t)$  and  $r \leq t$ , then  $u(r) \in \tau(r)$ . This allows to define the *boundary* of the tree at infinity by  $\partial \mathbf{T} \equiv \{u \in \mathbf{I} : \forall t < \infty, u(t) \in \tau(t)\}$ . In this way we identify each leaf of the Galton-Watson tree at time  $t$ ,  $i_k(t)$  with  $k \in \{1, \dots, n(t)\}$ , with some multi-label  $u^k(t) \in \tau(t)$ . We define the embedding  $\gamma$  by

$$\gamma(u(t)) \equiv \sum_{j=1}^{W(t)} u_j(t) e^{-t_j}. \quad (2.4)$$

For a given  $u$ , the function  $(\gamma(u(t)), t \in \mathbb{R}_+)$  describes a trajectory of a particle in  $\mathbb{R}_+$ , which converges to some point  $\gamma(u) \in \mathbb{R}_+$ , as  $t \uparrow \infty$ ,  $\mathbb{P}$ -a.s. Hence also the sets  $\gamma(\tau(t))$  converge, for any realisation of the tree, to some (random) set  $\gamma(\tau(\infty))$ .

Recall that in BBM there is also the position of the Brownian motion  $x_k(t)$  of the  $k$ -th particle at time  $t$ . Thus to any “particle” at time  $t$  we can now associate the position  $(\gamma(u^k(t)), x_k(t))$ , in  $\mathbb{R}_+ \times \mathbb{R}$ . Hoping that there will not be too much confusion, we will identify  $\gamma(u^k(t))$  with  $\gamma(x_k(t))$ .

**2.2. The critical martingale measure.** A key object is the derivative martingale  $Z_t$  defined in (1.2). Recall the following result proven in [28].

**Lemma 2.1.** *The limit  $Z := \lim_{t \rightarrow \infty} Z_t$  exists  $\mathbb{P}$ -a.s. and  $\min_{i \leq n(t)} (\sqrt{2}t - x_i(t)) \rightarrow \infty$  as  $t \rightarrow \infty$   $\mathbb{P}$ -a.s.*

For  $0 < r < t$  the truncated version

$$Z(v, r, t) := \sum_{j \leq n(t)} (\sqrt{2}t - x_j(t)) e^{\sqrt{2}(x_j(t) - \sqrt{2}t)} \mathbb{1}_{\{\gamma(x_j(r)) \leq v\}}, \quad v \in \mathbb{R}_+, \quad (2.5)$$

has been recently introduced in [12]. In particular, by [12, Lemma 3.2] for every  $v \in \mathbb{R}_+$  the limit

$$Z(v) := \lim_{r \uparrow \infty} \lim_{t \uparrow \infty} Z(v, r, t) \quad (2.6)$$

exists  $\mathbb{P}$ -a.s. Consider now the associated measures on  $\mathbb{R}_+$  given by

$$M_{r,t} := \sum_{j \leq n(t)} (\sqrt{2}t - x_j(t)) e^{\sqrt{2}(x_j(t) - \sqrt{2}t)} \delta_{\gamma(x_j(r))}, \quad (2.7)$$

and denote by  $M$  the Borel measure on  $\mathbb{R}_+$  defined via  $M([0, v]) = Z(v)$  for all  $v \in \mathbb{R}_+$ . Then, (2.6) implies that  $\mathbb{P}$ -a.s.

$$M = \lim_{r \uparrow \infty} \lim_{t \uparrow \infty} M_{r,t} \quad \text{weakly.} \quad (2.8)$$

By [12, Proposition 3.2]  $M$  is  $\mathbb{P}$ -a.s. non-atomic. Moreover, due to the recursive structure of the underlying GW-tree  $M$  is supported on some Cantor-like set  $\mathcal{X}$ .

**2.3. The PCAF of the truncated measure.** Let  $\Omega' := C([0, \infty), \mathbb{R})$  and let  $W = (W_t)_{t \geq 0}$  be the coordinate process on  $\Omega'$  and set  $\mathcal{G}_\infty^0 := \sigma(W_s; s < \infty)$  and  $\mathcal{G}_t^0 := \sigma(W_s; s \leq t)$ ,  $t \geq 0$ . Further, let  $\{P_x\}_{x \in \mathbb{R}}$  be the family of probability measures on  $(\Omega', \mathcal{G}_\infty^0)$  such that for each  $x \in \mathbb{R}$ ,  $W = (W_t)_{t \geq 0}$  under  $P_x$  is a one-dimensional Brownian motion starting at  $x$ . We denote by  $\{\mathcal{G}_t\}_{t \in [0, \infty]}$  the minimum completed admissible filtration for  $W$  and by  $L(W) = \{L_t^a(W), t \geq 0, a \in \mathbb{R}\}$  the random field of local times of  $W$ .

Now we set  $B_t := |W_t|$ ,  $t \geq 0$ , so that  $(\Omega', \mathcal{G}, (\mathcal{G}_t)_{t \geq 0}, (B_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}_+})$  is a reflected Brownian motion on  $\mathbb{R}_+$ . Then, the family  $L \equiv L(B) = \{L_t^a(B), t \geq 0, a \in \mathbb{R}_+\}$  of local times of  $B$  is given by

$$L_t^a \equiv L_t^a(B) = L_t^a(W) + L_t^{-a}(W), \quad t \geq 0, a \in \mathbb{R}_+ \quad (2.9)$$

(cf. [31, Exercise VI.1.17]).

**Proposition 2.2.** *For  $\mathbb{P}$ -a.e.  $\omega$ , there exists  $\tau_0 = \tau_0(\omega)$  such that for all  $t \geq \tau_0$  and  $0 \leq r < t$  the following hold.*

(i) *The unique PCAF of  $B$  with Revuz measure  $M_{r,t}$  is given by*

$$F_{r,t} : [0, \infty) \rightarrow [0, \infty) \quad s \mapsto \sum_{j=1}^{n(t)} (\sqrt{2}t - x_j(t)) e^{\sqrt{2}(x_j(t) - \sqrt{2}t)} L_s^{\gamma(x_j(r))}. \quad (2.10)$$

(ii) *For all  $x \in \mathbb{R}_+$ ,  $P_x$ -a.s.,  $F_{r,t}$  is continuous, increasing and satisfies  $F_{r,t}(0) = 0$  and  $\lim_{s \rightarrow \infty} F_{r,t}(s) = \infty$ .*

*Proof.* Recall that  $\min_{i \leq n(t)} (\sqrt{2}t - x_i(t)) \rightarrow \infty$   $\mathbb{P}$ -a.s. as  $t \rightarrow \infty$  by Lemma 2.1. Then, the statement follows immediately from Lemma B.3 and Lemma B.1.  $\square$



### 3. APPROXIMATION AND CONSTRUCTION OF THE PCAF OF $M$

This section is devoted to the proof of Theorem 1.2. In the following we will write

$$\mathbb{P}_x := \mathbb{P} \times P_x, \quad x \in \mathbb{R}_+, \quad (3.1)$$

for abbreviation. In a first step we establish the almost sure limit of  $F_{r,t}$  in  $t$  by using the fact that there is an explicit representation of that limit.

**Lemma 3.1.** *For any  $r > 0$  there exists a set  $\Lambda_1 = \Lambda_1(r) \subset \Omega \times \Omega'$  such that  $\mathbb{P}_x[\Lambda_1] = 1$  for all  $x \in \mathbb{R}_+$ , on which for any  $S > 0$ ,*

$$\sup_{s \leq S} |F_{r,t}(s) - F_r(s)| \rightarrow 0 \quad \text{as } t \uparrow \infty, \quad (3.2)$$

where

$$F_r(s) := \sum_{i=1}^{n(r)} e^{\sqrt{2}(x_i(r) - \sqrt{2}r)} Z^{(i)} L_s^{\gamma(x_i(r))}, \quad s \geq 0, \quad (3.3)$$

and  $Z^{(i)}, i \in \mathbb{N}$ , are i.i.d. copies of  $Z = \lim_{t \rightarrow \infty} Z_t$ .

*Proof.* Note that

$$\begin{aligned} F_{r,t}(s) &= \sum_{i=1}^{n(r)} e^{\sqrt{2}(x_i(r) - \sqrt{2}r)} L_s^{\gamma(x_i(r))} \left( \sum_{j=1}^{n^{(i)}(t-r)} \left( \sqrt{2}r - x_i(r) \right) e^{\sqrt{2}(x_j^{(i)}(t-r) - \sqrt{2}(t-r))} \right. \\ &\quad \left. + \sum_{j=1}^{n^{(i)}(t-r)} \left( \sqrt{2}(t-r) - x_j^{(i)}(t-r) \right) e^{\sqrt{2}(x_j^{(i)}(t-r) - \sqrt{2}(t-r))} \right) \\ &= \sum_{i=1}^{n(r)} L_s^{\gamma(x_i(r))} e^{\sqrt{2}(x_i(r) - \sqrt{2}r)} \left( \sqrt{2}r - x_i(r) \right) Y_{t-r}^{(i)} \\ &\quad + \sum_{i=1}^{n(r)} L_s^{\gamma(x_i(r))} e^{\sqrt{2}(x_i(r) - \sqrt{2}r)} Z_{t-r}^{(i)}, \end{aligned} \quad (3.4)$$

where for  $i \leq n(r)$  we denote by  $\{x_j^{(i)}(t-r), j \leq n^{(i)}(t-r)\}$  the particle positions of i.i.d. BBMs at time  $t-r$ . Moreover,  $Z_t^{(i)}, i \in \mathbb{N}$ , are i.i.d. copies of the derivative martingale, and  $Y_t^{(i)}$  are i.i.d. copies of the McKean martingale

$$Y_t := \sum_{j=1}^{n(t)} e^{\sqrt{2}x_j(t) - 2t}, \quad t \geq 0. \quad (3.5)$$

In [28] Lalley and Sellke proved that  $\mathbb{P}$ -a.s.  $\lim_{t \uparrow \infty} Y_t = 0$  and  $\lim_{t \uparrow \infty} Z_t = Z$  is a non-trivial random variable. This implies that there exists  $\Lambda_1 = \Lambda_1(r)$  such that

$\mathbb{P}_x[\Lambda_1] = 1$  for every  $x \in \mathbb{R}_+$ , on which for all  $s \geq 0$ ,

$$\lim_{t \uparrow \infty} F_{r,t}(s) = \sum_{i=1}^{n(r)} e^{\sqrt{2}(x_i(r) - \sqrt{2}r)} Z^{(i)} L_s^{\gamma(x_i(r))} =: F_r(s), \quad (3.6)$$

where  $Z^{(i)}$  are i.i.d. copies of  $Z$ . Moreover, note that  $L_s^{\gamma(x_i(r))}$  is monotone increasing in  $s$  for fixed  $r$ . On the other hand, recall that by Lemma 2.1 we have  $\min_{i \leq n(t)} (\sqrt{2}t - x_i(t)) \rightarrow +\infty$  as  $t \uparrow \infty$   $\mathbb{P}$ -a.s. Therefore, the part of the sum in (2.10) that involves negative terms (namely those for which  $x_i(t) > \sqrt{2}t$ ) converges to zero  $\mathbb{P}_x$ -a.s. The remaining part of the sum is increasing in  $s$ , and this implies that the limit  $F_r$  for  $t \uparrow \infty$  is monotone increasing in  $s$ . From the explicit form of the limit in (3.6) we obtain that the convergence is uniform in  $s \in [0, S]$ .  $\square$

For  $0 \leq r < t < \infty$  we define

$$Z_{r,t}^\gamma := \sum_{k=1}^{n(t)} (\sqrt{2}t - x_k(t)) e^{\sqrt{2}(x_k(t) - \sqrt{2}t)} \mathbb{1}_{\{|\gamma(x_k(t)) - \gamma(x_k(r))| \leq e^{-r/2}\}}. \quad (3.7)$$

Next we show that this thinned  $Z_{r,t}^\gamma$ , which only keeps track of particles whose values under  $\gamma$  do not change much over time, is close to the original measure  $Z_{r,t}$  in probability.

**Lemma 3.2.** *For any  $\varepsilon, \delta > 0$  there exist  $r_0 = r_0(\varepsilon)$  and  $t_0 = t_0(\varepsilon)$  such that for any  $r > r_0$  and  $t > 3r \vee t_0$ ,*

$$\mathbb{P}[|Z_t - Z_{r,t}^\gamma| > \delta] < \varepsilon. \quad (3.8)$$

*Proof.* For  $d \in \mathbb{R}$  and  $0 \leq r < t \leq u < \infty$  we define the event

$$\mathcal{A}_{r,t,u}(d) := \{\forall k \leq n(u) \text{ with } x_k(u) - m(u) > d : |\gamma(x_k(t)) - \gamma(x_k(r))| \leq e^{-r/2}\}. \quad (3.9)$$

Let  $\mathcal{F}_t := \sigma\{(x_k(s))_{1 \leq k \leq n(s)}, s \leq t\}$  and for  $\underline{A}, \bar{A} \in \mathbb{R}$  with  $\underline{A} < \bar{A}$  we set  $\phi(x) := \mathbb{1}_{[\underline{A}, \bar{A})}(x)$ . We observe that for any  $t > 0$  the martingale  $Z_t$  appeared in [3] (see Eq. (3.25) therein) in the  $\mathbb{P}$ -a.s. limit of

$$\begin{aligned} \lim_{u \uparrow \infty} \mathbb{E} \left[ \mathbb{E} \left[ \exp \left( - \sum_{i=1}^{n(u)} \phi(x_i(u) - m(u)) \right) \middle| \mathcal{F}_t \right] \right] \\ = c_t \mathbb{E} \left[ \exp \left( - C(e^{-\sqrt{2}\underline{A}} - e^{-\sqrt{2}\bar{A}}) Z_t \right) \right], \end{aligned} \quad (3.10)$$

where  $\lim_{t \uparrow \infty} c_t = 1$  and  $C$  is the same constant as in (1.1). Similarly, for any  $0 < r < t$  we can consider

$$\lim_{u \uparrow \infty} \mathbb{E} \left[ \exp \left( - \sum_{i=1}^{n(u)} \mathbb{1}_{\{|\gamma(x_i(t)) - \gamma(x_i(r))| \leq e^{-r/2}\}} \phi(x_i(u) - m(u)) \right) \right]. \quad (3.11)$$

Note that  $\mathbb{1}_{\{|\gamma(x_i(t)) - \gamma(x_i(r))| \leq e^{-r/2}\}}$  is measurable with respect to  $\mathcal{F}_t$ . Then, the limit in (3.11) can be treated similarly as the one in [3, Eq. (3.17)]. More precisely, by

repeating the analysis therein (where the sum in the analogue to [3, Eq. (3.19)] runs over particles with  $|\gamma(x_i(t)) - \gamma(x_i(r))| \leq e^{-r/2}$  only) we obtain

$$\begin{aligned} & \lim_{u \uparrow \infty} \mathbb{E} \left[ \mathbb{E} \left[ \exp \left( - \sum_{i=1}^{n(u)} \mathbb{1}_{\{|\gamma(x_i(t)) - \gamma(x_i(r))| \leq e^{-r/2}\}} \phi(x_i(u) - m(u)) \right) \middle| \mathcal{F}_t \right] \right] \\ &= c'_t \mathbb{E} \left[ \exp \left( - C(e^{-\sqrt{2}\underline{A}} - e^{-\sqrt{2}\overline{A}}) Z_{r,t}^\gamma \right) \right], \end{aligned} \quad (3.12)$$

where  $\lim_{t \uparrow \infty} c'_t = 1$ . Moreover, the expectations in (3.10) and (3.12) can be related as follows,

$$\begin{aligned} & \mathbb{E} \left[ \exp \left( - \sum_{i=1}^{n(u)} \mathbb{1}_{\{|\gamma(x_i(t)) - \gamma(x_i(r))| \leq e^{-r/2}\}} \phi(x_i(u) - m(u)) \right) \right] \\ & \geq \mathbb{E} \left[ \exp \left( - \sum_{i=1}^{n(u)} \phi(x_i(u) - m(u)) \right) \right] \geq \mathbb{E} \left[ \exp \left( - \sum_{i=1}^{n(u)} \phi(x_i(u) - m(u)) \right) \mathbb{1}_{\mathcal{A}_{r,t,u}(\underline{A})} \right] \\ &= \mathbb{E} \left[ \exp \left( - \sum_{i=1}^{n(u)} \mathbb{1}_{\{|\gamma(x_i(t)) - \gamma(x_i(r))| \leq e^{-r/2}\}} \phi(x_i(u) - m(u)) \right) \mathbb{1}_{\mathcal{A}_{r,t,u}(\underline{A})} \right] \\ & \geq \mathbb{E} \left[ \exp \left( - \sum_{i=1}^{n(u)} \mathbb{1}_{\{|\gamma(x_i(t)) - \gamma(x_i(r))| \leq e^{-r/2}\}} \phi(x_i(u) - m(u)) \right) \right] - \mathbb{P} \left[ (\mathcal{A}_{r,t,u}(\underline{A}))^c \right]. \end{aligned} \quad (3.13)$$

Let  $\varepsilon > 0$ . By [12, Lemma 4.2] there exist  $r_0(\varepsilon)$  and  $t_0(\varepsilon)$  such that for all  $t \geq t_0(\varepsilon)$  and  $r > r_0(\varepsilon)$ ,

$$\lim_{u \uparrow \infty} \mathbb{P} \left[ (\mathcal{A}_{r,t,u}(\underline{A}))^c \right] < \varepsilon. \quad (3.14)$$

Hence, by combining (3.13) with (3.10) and (3.12) we get

$$\begin{aligned} & c'_t \mathbb{E} \left[ \exp \left( - C(e^{-\sqrt{2}\underline{A}} - e^{-\sqrt{2}\overline{A}}) Z_{r,t}^\gamma \right) \right] - \varepsilon \\ & \leq c_t \mathbb{E} \left[ \exp \left( - C(e^{-\sqrt{2}\underline{A}} - e^{-\sqrt{2}\overline{A}}) Z_t \right) \right] \\ & \leq c'_t \mathbb{E} \left[ \exp \left( - C(e^{-\sqrt{2}\underline{A}} - e^{-\sqrt{2}\overline{A}}) Z_{r,t}^\gamma \right) \right]. \end{aligned} \quad (3.15)$$

Recall that  $Z_t \rightarrow Z$   $\mathbb{P}$ -a.s. as  $t \rightarrow \infty$  (cf. [28]), where  $Z$  is  $\mathbb{P}$ -a.s. positive, and  $\lim_{t \uparrow \infty} c_t = \lim_{t \uparrow \infty} c'_t = 1$ . Hence, for all  $t$  and  $r$  sufficiently large,

$$\mathbb{P} \left[ \left| \exp \left( - C(e^{-\sqrt{2}\underline{A}} - e^{-\sqrt{2}\overline{A}}) Z_{r,t}^\gamma \right) - \exp \left( - C(e^{-\sqrt{2}\underline{A}} - e^{-\sqrt{2}\overline{A}}) Z_t \right) \right| > \delta \right] < \varepsilon. \quad (3.16)$$

The claim now follows from the continuous mapping theorem since  $\exp$  is injective and continuous.  $\square$

In the next lemma we lift the statement of Lemma 3.2 on the level of the associated PCAFs, meaning that with high probability the PCAF  $F_{r,t}$  and its thinned version  $F_{r,t}^\gamma$  are close to each other.

**Lemma 3.3.** *For any  $\varepsilon, \delta > 0$  and any  $S > 0$  there exist  $r_1 = r_1(\varepsilon, \delta, S)$  and  $t_1 = t_1(\varepsilon)$  such that for all  $r > r_1$  and  $t > 3r \vee t_1$  the following holds. There exists a set  $\Lambda_2 = \Lambda_2(\varepsilon, \delta, S, r, t) \subset \Omega \times \Omega'$  with  $\mathbb{P}_x[\Lambda_2^c] < \varepsilon$  for all  $x \in \mathbb{R}_+$  such that*

$$\sup_{s \leq S} |F_{r,t}(s) - F_{r,t}^\gamma(s)| \leq \delta \quad \text{on } \Lambda_2, \quad (3.17)$$

where  $F_{r,t}^\gamma : [0, \infty) \rightarrow [0, \infty)$  is defined by

$$F_{r,t}^\gamma(s) := \sum_{j=1}^{n(t)} (\sqrt{2}t - x_j(t)) e^{\sqrt{2}(x_j(t) - \sqrt{2}t)} \mathbb{1}_{\{|\gamma(x_j(t)) - \gamma(x_j(r))| \leq e^{-r/2}\}} L_s^{\gamma(x_j(r))}. \quad (3.18)$$

*Proof.* Recall that by Lemma 2.1 for  $\mathbb{P}$ -a.e.  $\omega$  there exists  $\tau_0 = \tau_0(\omega)$  such that  $\min_{i \leq n(t)} (\sqrt{2}t - x_i(t)) > 0$  for all  $t > \tau_0$ . Further, Lemma B.2 gives that for any  $\varepsilon > 0$  there exists  $\lambda = \lambda(\varepsilon, S)$  such that for all  $x \in \mathbb{R}_+$ ,

$$P_x \left[ \sup_{a \in \mathbb{R}_+} L_S^a > \lambda \right] < \varepsilon. \quad (3.19)$$

Together with Lemma 3.2 this implies that there exist  $r_0 = r_0(\varepsilon, \delta, S)$  and  $t_0 = t_0(\varepsilon, \delta, S)$  such that for all  $r > r_0$  and  $t > 3r \vee t_0$  there is a set  $\Lambda_2 = \Lambda_2(\varepsilon, \delta, S, r, t)$  with  $\mathbb{P}_x[\Lambda_2^c] < \varepsilon$  for all  $x \in \mathbb{R}_+$  on which

- $t > \tau_0$ ,
- $\sup_{a \in \mathbb{R}_+} L_S^a \leq \lambda$ ,
- $|Z_t - Z_{r,t}^\gamma| \leq \delta/\lambda$ .

Finally, note that on the set  $\Lambda_2$ ,

$$\begin{aligned} \sup_{s \leq S} |F_{r,t}(s) - F_{r,t}^\gamma(s)| &\leq |Z_t - Z_{r,t}^\gamma| \sup_{s \leq S} \max_{k \leq n(t)} L_s^{\gamma(x_k(r))} \\ &\leq |Z_t - Z_{r,t}^\gamma| \sup_{a \in \mathbb{R}_+} L_S^a \leq \delta, \end{aligned} \quad (3.20)$$

which completes the proof.  $\square$

The next proposition verifies Cauchy's convergence criterion for  $F_r$  as  $r \rightarrow \infty$  in  $\mathbb{P}_x$ -probability, which already implies the existence of the limit in  $\mathbb{P}_x$ -probability.

**Proposition 3.4.** *For any  $\varepsilon, \delta > 0$  and any  $S > 0$  there exists  $r_2 = r_2(\varepsilon, \delta, S)$  such that for all  $r \geq r' > r_2$  the following holds. There exists a set  $\Lambda_3 = \Lambda_3(\varepsilon, \delta, S, r, r') \subset \Omega \times \Omega'$  with  $\mathbb{P}_x[\Lambda_3^c] < \varepsilon$  for all  $x \in \mathbb{R}_+$  such that*

$$\sup_{s \leq S} |F_r(s) - F_{r'}(s)| \leq \delta \quad \text{on } \Lambda_3. \quad (3.21)$$

*Proof.* First, the triangle inequality gives that for any  $t \geq r \geq r' > 0$ ,

$$\begin{aligned} |F_r(s) - F_{r'}(s)| &\leq |F_r(s) - F_{r,t}(s)| + |F_{r,t}(s) - F_{r,t}^\gamma(s)| + |F_{r,t}^\gamma(s) - F_{r',t}^\gamma(s)| \\ &\quad + |F_{r',t}^\gamma(s) - F_{r'}(s)| + |F_{r',t}^\gamma(s) - F_{r',t}(s)|. \end{aligned} \quad (3.22)$$

In view of Lemma 3.1 and Lemma 3.3 for all  $r \geq r' > r_1(\varepsilon/2, \delta/4, S)$  and all  $t \geq t_2$  for a suitable  $t_2 = t_2(\varepsilon, r, r')$  there exists a set  $\Lambda' = \Lambda'(\varepsilon, \delta, S, r, r', t)$  with

$\mathbb{P}_x [(\Lambda')^c] < \varepsilon$  on which the sum of the first two terms and the last two terms in the right hand side of (3.22) is bounded from above by  $\delta$  uniformly in  $s \in [0, S]$ .

Hence, it suffices to show that for any  $\varepsilon, \delta > 0$  and  $S > 0$  there exists  $r_3 = r_3(\varepsilon, \delta, S)$  such that for all  $r \geq r' > r_3$  and all  $t \geq t_3$  for a suitable  $t_3 = t_3(\varepsilon, r, r')$  there exists a set  $\Lambda'' = \Lambda''(\varepsilon, \delta, S, r, r', t)$  with  $\mathbb{P}_x [(\Lambda'')^c] < \varepsilon$  on which

$$\sup_{s \leq S} |F_{r,t}^\gamma(s) - F_{r',t}^\gamma(s)| \leq \delta. \quad (3.23)$$

We now show (3.23). Recall the definition of the event  $\mathcal{A}_{r,t,u}$  in (3.9). Using (3.14) based on [12, Lemma 4.2] we have that for all  $r \geq r' > r_0(\varepsilon)$  and  $t \geq t_0(\varepsilon)$ ,

$$\mathbb{P}[(\mathcal{A}_{r,t,u})^c \cup (\mathcal{A}_{r',t,u})^c] < \varepsilon \quad (3.24)$$

for some  $u = u(r, r', t)$ . Now, for some suitable  $\kappa = \kappa(\varepsilon, S) > 0$ ,  $z = z(\varepsilon) > 0$  and  $r_3 = r_3(\varepsilon, S)$  Lemma 2.1 and Lemma B.1 (ii) give for all  $r \geq r' > r_3$  and all  $t \geq t_3$  for some  $t_3 = t_3(\varepsilon, r, r')$  the existence of a set  $\Lambda'' = \Lambda''(\varepsilon, S, r, r', t)$  with  $\mathbb{P}_x [(\Lambda'')^c] < \varepsilon$  on which

- the events  $\mathcal{A}_{r',t,u}$  and  $\mathcal{A}_{r,t,u}$  hold, in particular

$$|\gamma(x_j(r)) - \gamma(x_j(r'))| \leq e^{-r/2} + e^{-r'/2}, \quad \forall j \leq n(t), \quad (3.25)$$

- $Z_t \leq 2Z \leq 2z$ ,
- the Hölder-continuity of the local times in Lemma B.1 (ii) holds and the Hölder constant  $C_1 = C_1(\omega', S, \alpha)$  appearing in (B.1) satisfies  $C_1 \leq \kappa$ .

Note that on the set  $\Lambda''$ ,

$$\begin{aligned} \sup_{s \leq S} |F_{r,t}^\gamma(s) - F_{r',t}^\gamma(s)| &\leq \sum_{j \leq n(t)} (\sqrt{2}t - x_j(t)) e^{\sqrt{2}(x_j(t) - \sqrt{2}t)} \sup_{s \leq S} |L_s^{\gamma(x_j(r))} - L_s^{\gamma(x_j(r'))}| \\ &\leq C_1 Z_t \max_{j \leq n(t)} |\gamma(x_j(r)) - \gamma(x_j(r'))|^\alpha \\ &\leq C_1 Z_t (e^{-r/2} + e^{-r'/2})^\alpha \\ &\leq 2\kappa z (e^{-r/2} + e^{-r'/2})^\alpha. \end{aligned} \quad (3.26)$$

Finally, possibly after choosing a larger  $r_3$  depending also on  $\delta$ , we obtain that  $2\kappa z (e^{-r/2} + e^{-r'/2})^\alpha \leq \delta$  for all  $r \geq r' > r_3$  and (3.23) follows.  $\square$

For any set  $A \in \mathcal{F} \otimes \mathcal{G}_\infty^0$  we will write

$$A^\omega := \{\omega' \in \Omega' : (\omega, \omega') \in A\}, \quad \omega \in \Omega. \quad (3.27)$$

From Proposition 3.4 we now derive the  $\mathbb{P}_x$ -a.s. convergence of  $F_r$  along subsequences using a Borel Cantelli argument.

**Proposition 3.5.** *For any  $S > 0$  and any sequence  $(r_n)_{n \in \mathbb{N}}$  satisfying  $r_n \uparrow \infty$  there exist a subsequence  $(r_{n_k})_{k \in \mathbb{N}}$  and a set  $\Lambda_4 = \Lambda_4(S, (r_{n_k})_{k \in \mathbb{N}}) \subset \Omega \times \Omega'$ , such that*

$\mathbb{P}_x[\Lambda_4] = 1$  for all  $x \in \mathbb{R}_+$ , and a continuous and increasing functional  $F : [0, \infty) \rightarrow [0, \infty)$  such that

$$\lim_{k \rightarrow \infty} \sup_{s \leq S} |F_{r_{n_k}}(s) - F(s)| = 0 \quad \text{on } \Lambda_4. \quad (3.28)$$

For  $\mathbb{P}$ -a.e.  $\omega$ , the functional  $F$  is an increasing PCAF with defining set  $\Lambda_4^\omega$  and satisfies  $F(0) = 0$  and  $\lim_{s \rightarrow \infty} F(s) = \infty$ .

*Proof.* By Proposition 3.4 for any  $S > 0$  and  $k \in \mathbb{N}$  we can choose  $r_{n_k}$  such that for all  $k' \geq k$  the following holds. The set  $\Lambda_3 \equiv \Lambda_3(2^{-k}, 2^{-k}, S, r_{n_k}, r_{n_{k'}})$  satisfies  $\mathbb{P}_x[\Lambda_3^c] < 2^{-k}$  for all  $x \in \mathbb{R}_+$  and we have

$$\sup_{s \leq S} |F_{r_{n_{k'}}}(s) - F_{r_{n_k}}(s)| \leq 2^{-k} \quad \text{on } \Lambda_3. \quad (3.29)$$

In particular, for all  $k \in \mathbb{N}$  there exists a set  $N_k = N_k(S) \subset \Omega \times \Omega'$  with  $\mathbb{P}_x[N_k] < 2^{-k}$  for all  $x \in \mathbb{R}_+$  such that

$$\sup_{s \leq S} |F_{r_{n_{k+1}}}(s) - F_{r_{n_k}}(s)| \leq 2^{-k} \quad \text{on } N_k^c. \quad (3.30)$$

Note that for all  $x \in \mathbb{R}_+$ ,

$$\sum_{k=1}^{\infty} \mathbb{P}_x[N_k] \leq 1 < \infty, \quad (3.31)$$

and, setting  $N := \limsup_k N_k$ , we obtain  $\mathbb{P}_x[N] = 0$  for all  $x \in \mathbb{R}_+$ . Note that on the set  $\Lambda_4 := N^c$  the inequality  $\sup_{s \leq S} |F_{r_{n_{k+1}}}(s) - F_{r_{n_k}}(s)| > 2^{-k}$  can only be satisfied for at most finitely many  $k \in \mathbb{N}$ , so that

$$\sum_{k=1}^{\infty} \sup_{s \leq S} |F_{r_{n_{k+1}}}(s) - F_{r_{n_k}}(s)| < \infty, \quad (3.32)$$

which implies that on  $\Lambda_4$  the sequence  $(F_{r_{n_k}})_k$  is a Cauchy sequence w.r.t. the sup-norm and thus it converges to some functional  $F$ . The properties of the limit follow from the almost sure convergence and Proposition 2.2.  $\square$

Note that the limiting functional  $F$  in Proposition 3.5 might depend a-priori on the chosen sequence and subsequence. In order to complete the proof of Theorem 1.2 we give an explicit representation for  $F$  and identify it as the unique PCAF with Revuz measure  $M$ . We start with a preparatory lemma.

**Lemma 3.6.** *For  $\mathbb{P}$ -a.e.  $\omega$  there exists  $r_0 = r_0(\omega)$  such that the following holds. For any  $x \in \mathbb{R}_+$ ,  $S > 0$  and any bounded Borel measurable function  $f : \mathbb{R}_+ \rightarrow [0, \infty)$  the family  $\{\int_0^S f(B_s) dF_{r,t}(s)\}_{t \geq r \geq r_0}$  is uniformly  $P_x$ -integrable.*

*Proof.* Recall that  $\mathbb{P}$ -a.s.  $Z_t \rightarrow Z$  (cf. Lemma 2.1), so for  $\mathbb{P}$ -a.e.  $\omega$  there exists  $r_0 = r_0(\omega)$  such that  $Z_t \leq 2Z$  for all  $t \geq r_0$ . It suffices to prove that  $\mathbb{P}$ -a.s. for any  $x \in \mathbb{R}_+$ ,

$$\sup_{t \geq r \geq r_0} E_x \left[ \left| \int_0^S f(B_s) dF_{r,t}(s) \right| \right] < \infty. \quad (3.33)$$

Note that

$$\int_0^S f(B_s) dF_{r,t}(s) = \sum_{j \leq n(t)} (\sqrt{2}t - x_j(t)) e^{\sqrt{2}(x_j(t) - \sqrt{2}t)} f(\gamma(x_j(r))) L_S^{\gamma(x_j(r))}, \quad (3.34)$$

so that

$$E_x \left[ \left| \int_0^S f(B_s) dF_{r,t}(s) \right| \right] \leq \|f\|_\infty |Z_t| E_x \left[ \sup_{a \in \mathbb{R}_+} L_S^a \right] \leq 2\|f\|_\infty Z E_x \left[ \sup_{a \in \mathbb{R}_+} L_S^a \right], \quad (3.35)$$

and (3.33) follows from Lemma B.2.  $\square$

**Proposition 3.7.** *Any limiting functional  $F$  in Proposition 3.5 is  $\mathbb{P}$ -a.s. equivalent to the unique PCAF with Revuz measure  $M$ .*

*Proof.* Recall that only the empty set is polar for  $B$ . In particular, the measure  $M$  does trivially not charge polar sets, so by general theory (see e.g. [16, Theorem 4.1.1]) the PCAF with Revuz measure  $M$  is (up to equivalence) unique. Thus, we need show that any limiting functional  $F$  in Proposition 3.5 is  $\mathbb{P}$ -a.s. in Revuz correspondence with  $M$ . In view of (A.3) it suffices to prove that  $\mathbb{P}$ -a.s.

$$\int_{\mathbb{R}_+} f(a) M(da) = \int_{\mathbb{R}_+} E_x \left[ \int_0^1 f(B_s) dF(s) \right] dx \quad (3.36)$$

for any non-negative Borel function  $f : \mathbb{R}_+ \rightarrow [0, \infty]$ . By a monotone class argument it is enough to consider continuous functions  $f$  with compact support in  $\mathbb{R}_+$ . Note that  $E_x \left[ \int_0^1 f(B_s) dL_s^a \right] = f(a) E_x [L_1^a]$  for any  $a \in \mathbb{R}_+$  and therefore

$$E_x \left[ \int_0^1 f(B_s) dF_{r,t}(s) \right] = \int_{\mathbb{R}_+} f(a) E_x [L_1^a] M_{r,t}(da). \quad (3.37)$$

By Lemma B.2 we have  $\sup_{a \in \mathbb{R}_+} E_x [L_1^a] < \infty$  and together with Lemma B.1 this implies that the mapping  $a \mapsto f(a) E_x [L_1^a]$  is bounded and continuous on  $\mathbb{R}_+$ . Furthermore, by Lemma 3.1 and Proposition 3.5 the sequence  $(dF_{r,t})$  converges weakly to  $dF$  on  $[0, 1]$ ,  $\mathbb{P}_x$ -a.s. along the chosen subsequence. We take limits in  $t$  and  $r$  on both sides of (3.37), where we use Lemma 3.6 for the left hand side and the weak convergence of  $M_{r,t}$  towards  $M$  for the right hand side, and obtain

$$E_x \left[ \int_0^1 f(B_s) dF(s) \right] = \int_{\mathbb{R}_+} f(a) E_x [L_1^a] M(da). \quad (3.38)$$

Finally, by integrating both sides over  $x \in \mathbb{R}_+$  and using Fubini's theorem and Lemma B.4 we get (3.36).  $\square$

**Proposition 3.8.** *For any  $x \in \mathbb{R}_+$  the functional  $F$  in Proposition 3.5 is  $\mathbb{P}_x$ -a.s. explicitly given by*

$$F(S) = \int_{\mathbb{R}_+} L_S^a M(da), \quad \forall S \geq 0. \quad (3.39)$$

*Proof.* By an approximation argument it suffices to show that  $\mathbb{P}$ -a.s. for any continuous function  $f$  with compact support in  $\mathbb{R}_+$ ,

$$E_x \left[ \left| \int_0^S f(B_s) dF(s) - \int_{\mathbb{R}_+} f(a) L_S^a M(da) \right| \right] = 0. \quad (3.40)$$

Recall that  $P_x$ -a.s.  $\int_0^S f(B_s) dL_s^a = f(a) L_S^a$  for any  $a \in \mathbb{R}_+$ , so that for any  $r < t$ ,

$$\int_0^S f(B_s) dF_{r,t}(s) = \int_{\mathbb{R}_+} f(a) L_S^a M_{r,t}(da). \quad (3.41)$$

Hence, the expectation in (3.40) is bounded from above by

$$\begin{aligned} & E_x \left[ \left| \int_0^S f(B_s) dF(s) - \int_0^S f(B_s) dF_{r,t}(s) \right| \right] \\ & + E_x \left[ \left| \int_{\mathbb{R}_+} f(a) L_S^a M_{r,t}(da) - \int_{\mathbb{R}_+} f(a) L_S^a M(da) \right| \right]. \end{aligned} \quad (3.42)$$

By Lemma 3.1 and Proposition 3.5 the sequence  $(dF_{r,t})$  converges weakly to  $dF$  on  $[0, S]$ ,  $\mathbb{P}_x$ -a.s. along the chosen subsequence. Using Lemma 3.6 we obtain that the first summand in (3.42) converges to zero as first  $t$  and then  $r$  tend to infinity. The second summand can be decomposed into

$$\begin{aligned} & E_x \left[ \left| \int_{\mathbb{R}_+} f(a) L_S^a M_{r,t}(da) - \int_{\mathbb{R}_+} f(a) L_S^a M(da) \right| \mathbb{1}_{\{\sup_a L_S^a \leq \lambda\}} \right] \\ & + E_x \left[ \left| \int_{\mathbb{R}_+} f(a) L_S^a M_{r,t}(da) - \int_{\mathbb{R}_+} f(a) L_S^a M(da) \right| \mathbb{1}_{\{\sup_a L_S^a > \lambda\}} \right] \end{aligned} \quad (3.43)$$

for any  $\lambda > 0$ . On the event  $\{\sup_a L_S^a \leq \lambda\}$  the mapping  $a \mapsto f(a) L_S^a$  is bounded and continuous on  $\mathbb{R}_+$  and since  $M_{r,t}$  converges weakly towards  $M$  the first term in (3.43) converges to zero as  $t$  and  $r$  tend to infinity. By the Cauchy Schwarz inequality the second term in (3.43) can be bounded from above by

$$\left( \int_{\mathbb{R}_+} f(a) M_{r,t}(da) + \int_{\mathbb{R}_+} f(a) M(da) \right) E_x \left[ \left( \sup_a L_S^a \right)^2 \right]^{\frac{1}{2}} P_x \left[ \sup_a L_S^a > \lambda \right]^{\frac{1}{2}}. \quad (3.44)$$

Again by the weak convergence of the measures  $M_{r,t}$  the first term in (3.44) remains finite as  $t$  and  $r$  tend to infinity. Finally, using Lemma B.2 and taking  $\lambda \rightarrow \infty$  we complete the proof.  $\square$

*Proof of Theorem 1.2.* The  $\mathbb{P}_x$ -a.s. limit  $F_r = \lim_{t \rightarrow \infty} F_{r,t}$  has been established in Lemma 3.1, so it remains to show that  $\lim_{r \rightarrow \infty} F_r = F$  in  $\mathbb{P}_x$ -probability, where  $F$  explicitly given by (1.9) is the unique PCAF having Revuz measure  $M$ . The conjunction of Propositions 3.5, 3.7 and 3.8 gives this convergence along a suitable subsequence  $(r_{n_k})$ . Since for any  $\delta > 0$ ,

$$\begin{aligned} & \mathbb{P}_x \left[ \sup_{s \leq S} |F_r(s) - F(s)| > \delta \right] \\ & \leq \mathbb{P}_x \left[ \sup_{s \leq S} |F_r(s) - F_{r_{n_k}}(s)| > \frac{\delta}{2} \right] + \mathbb{P}_x \left[ \sup_{s \leq S} |F_{r_{n_k}}(s) - F(s)| > \frac{\delta}{2} \right], \end{aligned} \quad (3.45)$$



the claim follows from Proposition 3.4 and Proposition 3.5.  $\square$

#### 4. PROPERTIES AND APPROXIMATION OF THE PROCESS

**4.1. First properties.** Recall that the process  $\mathcal{B}$  is defined as the time-changed Brownian motion

$$\mathcal{B}(s) := B_{F^{-1}(s)}, \quad s \geq 0, \quad (4.1)$$

where  $F$  is the PCAF in Theorem 1.2. First, we observe that the continuity of  $F$  ensures that the process  $\mathcal{B}$  does not get stuck anywhere in the state space, and since  $\lim_{s \rightarrow \infty} F(s) = \infty$   $\mathbb{P}_x$ -a.s. the process  $\mathcal{B}$  does not explode in finite time. However,  $F$  is not strictly increasing so that jumps occur.

More precisely, by the general theory of time changes of Markov processes we have the following properties of  $\mathcal{B}$ . First, in view of [24, Theorems A.2.12]  $\mathcal{B}$  is a right-continuous strong Markov process on  $\mathcal{X} := \text{supp } M$  and by [16, Proposition A.3.8] we have  $\mathbb{P}$ -a.s.

$$P_x[\mathcal{B}(s) \in \tilde{\mathcal{X}}, \forall s \geq 0] = 1, \quad \forall x \in \mathcal{X}, \quad (4.2)$$

where  $\tilde{\mathcal{X}}$  denotes the support of the PCAF  $F$ , i.e.

$$\tilde{\mathcal{X}} := \{x \in \mathbb{R}_+ : P_x[R = 0] = 1\} \quad \text{with} \quad R := \inf\{s > 0 : F_s > 0\}. \quad (4.3)$$

By general theory (cf. [24, Section 5.1]) we have  $\tilde{\mathcal{X}} \subseteq \mathcal{X}$  (recall that only the empty set is polar) and  $\mathcal{X} \setminus \tilde{\mathcal{X}}$  has  $M$ -measure zero.

Furthermore, by [24, Theorem 6.2.3] the process  $\mathcal{B}$  is recurrent and by [24, Theorem 6.2.1 (i)] the transition function  $(P_s)_{s>0}$  of  $\mathcal{B}$  given by

$$P_s f(x) := E_x[f(\mathcal{B}(s))], \quad s > 0, x \in \mathcal{X}, f \in L^2(\mathcal{X}, M), \quad (4.4)$$

determines a strongly continuous semigroup and is  $M$ -symmetric, i.e. it satisfies

$$\int_{\mathcal{X}} P_s f \cdot g \, dM = \int_{\mathcal{X}} f \cdot P_s g \, dM \quad (4.5)$$

for all Borel measurable functions  $f, g : \mathcal{X} \rightarrow [0, \infty]$ .

**4.2. The Dirichlet form.** We can apply the general theory of Dirichlet forms to obtain a more precise description of the Dirichlet form associated with  $\mathcal{B}$ . For  $D = (0, \infty)$  denote by  $H^1(D)$  the standard Sobolev space, that is

$$H^1(D) = \{f \in L^2(D, dx) : f' \in L^2(D, dx)\}, \quad (4.6)$$

where the derivatives are in the distributional sense. On  $H^1(D)$  we define the form

$$\mathcal{E}(f, g) = \frac{1}{2} \int_{\mathbb{R}_+} f' \cdot g' \, dx. \quad (4.7)$$

Recall that  $(\mathcal{E}, H^1(D))$  can be regarded as a regular Dirichlet form on  $L^2(\mathbb{R}_+)$  and the associated process is the reflected Brownian motion  $B$  on  $\mathbb{R}_+$ . By  $H_e^1(\mathbb{R}_+)$  we denote the extended Dirichlet space, that is the set of  $dx$ -equivalence classes of Borel measurable functions  $f$  on  $\mathbb{R}_+$  such that  $\lim_{n \rightarrow \infty} f_n = f \in \mathbb{R}$   $dx$ -a.e. for

some  $(f_n)_{n \geq 1} \subset H^1(\mathbb{R}_+)$  satisfying  $\lim_{k,l \rightarrow \infty} \mathcal{E}(f_k - f_l, f_k - f_l) = 0$ . By [16, Theorem 2.2.13] we have the following identification of  $H_e^1(D)$ :

$$H_e^1(D) = \{f \in L_{\text{loc}}^2(D, dx) : f' \in L^2(D, dx)\}. \quad (4.8)$$

Recall that  $\mathcal{X}$  denotes the support of the random measure  $M$ . We define the hitting distribution

$$H_{\mathcal{X}}f(x) := E_x[f(B_{\sigma_{\mathcal{X}}})], \quad x \in \mathbb{R}_+, \quad (4.9)$$

with  $\sigma_{\mathcal{X}} := \inf\{t > 0 : B_t \in \mathcal{X}\}$  for any non-negative Borel function  $f$  on  $\mathbb{R}_+$ . Note that the function  $H_{\mathcal{X}}f$  is uniquely determined by the restriction of  $f$  to the set  $\mathcal{X}$ . Further, by [16, Theorem 3.4.8], we have  $H_{\mathcal{X}}f \in H_e^1(D)$  and by [24, Lemma 6.2.1]  $H_{\mathcal{X}}f = H_{\mathcal{X}}g$  whenever  $f = g$   $M$ -a.e. for any  $f, g \in H_e^1(D)$ . Therefore it makes sense to define the symmetric form  $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$  on  $L^2(\mathcal{X}, M)$  by

$$\begin{cases} \hat{\mathcal{F}} := \{\varphi \in L^2(\mathcal{X}, M) : \varphi = f \text{ } M\text{-a.e. for some } f \in H_e^1(D)\}, \\ \hat{\mathcal{E}}(\varphi, \varphi) := \mathcal{E}(H_{\mathcal{X}}f, H_{\mathcal{X}}f), \quad \varphi \in \hat{\mathcal{F}}, \varphi = f \text{ } M\text{-a.e., } f \in H_e^1(D). \end{cases} \quad (4.10)$$

By [24, Theorem 6.2.1]  $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$  is the regular Dirichlet form on  $L^2(\mathcal{X}; M)$  associated with the process  $\mathcal{B}$ . Since  $\mathcal{X}$  has Lebesgue measure zero, it follows from the Beurling-Deny representation formula for  $\hat{\mathcal{E}}$  (see [16, Theorem 5.5.9]) that  $\mathcal{B}$  has no diffusive part and is therefore a pure jump process.

**4.3. Approximation of the process.** For any  $0 < r < t$  we define

$$\mathcal{B}_{r,t}(s) := B_{F_{r,t}^{-1}(s)}, \quad s \geq 0, \quad (4.11)$$

where  $F_{r,t}^{-1}$  denotes the right-continuous inverse of  $F_{r,t}$ . Further, let  $D([0, \infty), \mathbb{R}_+)$  be the space of  $\mathbb{R}_+$ -valued càdlàg paths. We denote by  $d_{J_1}$  and  $d_{M_1}$  the metric on  $D([0, \infty), \mathbb{R}_+)$  (or  $D([0, S], \mathbb{R}_+)$ ) w.r.t. Skorohod  $J_1$ - and  $M_1$ -topology, respectively. We refer to [37, Chapter 3] for the precise definitions. Further, let

$$D_{\uparrow}([0, \infty), \mathbb{R}_+) := \{w \in D([0, \infty), \mathbb{R}_+) : w \text{ non-decreasing, } w(0) = w^{-1}(0) = 0\},$$

where  $w^{-1}$  denotes the right-continuous inverse of  $w$ . Finally, we set

$$L_{\text{loc}}^1 := \left\{w \in D([0, \infty), \mathbb{R}_+) : \int_0^S |w(s)| ds < \infty \text{ for all } S \geq 0\right\},$$

equipped with the topology induced by supposing

$$w_n \rightarrow w \text{ if and only if } \int_0^S |w_n(s) - w(s)| ds \rightarrow 0 \text{ for all } S \geq 0.$$

Note that the  $L_{\text{loc}}^1$ -topology extends both the  $J_1$ - and the  $M_1$ -topology since it allows excursions in the approximating processes which are not present in the limit process provided they are of negligible  $L^1$ -magnitude (cf. [18, Remark 1.3]).

**Theorem 4.1.** *Under the annealed law  $\int P_0(\cdot) d\mathbb{P}$ ,*

$$\lim_{r \uparrow \infty} \lim_{t \uparrow \infty} \mathcal{B}_{r,t} = \mathcal{B} \quad (4.12)$$

*in distribution on  $L^1_{\text{loc}}$ , that is for all bounded continuous functions  $f$  on  $L^1_{\text{loc}}$  we have*

$$\lim_{r \uparrow \infty} \lim_{t \uparrow \infty} \mathbb{E}[E_0[f(\mathcal{B}_{r,t})]] = \mathbb{E}[E_0[f(\mathcal{B})]]. \quad (4.13)$$

*Remark 4.2.* Since the measures  $M_{r,t}$  and  $M$  do not have full support and  $F_{r,t}^{-1}$  and  $F^{-1}$  have discontinuities, the locally uniform convergence of the functionals  $F_{r,t}$  only implies the  $M_1$ -convergence of their inverses. In such a situation the composition mapping is only continuous in the  $L^1_{\text{loc}}$ -topology (see Lemma 4.3 below), which is why we only obtain the approximation in Theorem 4.1 in the coarser  $L^1_{\text{loc}}$ -topology. We refer to [17, Corollary 1.5 (b)] for a similar result and to [18, 23, 29] for examples of convergence results for trap models in the  $L^1_{\text{loc}}$ -topology (or slight modifications of it).

Before we prove Theorem 4.1 we recall some facts about the continuity of the inverse and the composition mapping on the space of càdlàg paths.

**Lemma 4.3.** (i) *For any  $w_1, w_2 \in D([0, S], \mathbb{R}_+)$ ,*

$$d_{M_1}(w_1, w_2) \leq d_{J_1}(w_1, w_2) \leq \sup_{s \in [0, S]} |w_1(s) - w_2(s)|. \quad (4.14)$$

(ii) *Let  $(a_n)$  be a sequence in  $D_{\uparrow}([0, \infty), \mathbb{R}_+)$  such that  $a_n \rightarrow a$  in  $M_1$ -topology for some  $a \in D_{\uparrow}([0, \infty), \mathbb{R}_+)$ . Then,  $a_n^{-1} \rightarrow a^{-1}$  in  $M_1$ -topology, where  $a_n^{-1}$  and  $a^{-1}$  denote the right-continuous inverses of  $a_n$  and  $a$ , respectively.*

(iii) *Let  $(a_n) \subset D_{\uparrow}([0, \infty), \mathbb{R}_+)$  and  $(w_n) \subset D([0, \infty), \mathbb{R}_+)$  such that  $a_n \rightarrow a$  in  $M_1$ -topology for some  $a \in D_{\uparrow}([0, \infty), \mathbb{R}_+)$  and  $w_n \rightarrow w$  in  $J_1$ -topology for some  $w \in C([0, \infty), \mathbb{R}_+)$ . Then,  $w_n \circ a_n \rightarrow w \circ a$  in  $L^1_{\text{loc}}$ -topology.*

*Proof.* For the first inequality in (4.14) we refer to [37, Theorem 12.3.2] and the second inequality is immediate from the definition of the  $J_1$ -metric. Statement (ii) follows from the continuity of the inverse mapping in  $D((0, \infty), \mathbb{R}_+)$ , see [37, Corollary 13.6.5]; the continuity at zero is clear since  $a_n^{-1}(0) = 0$  and  $a^{-1}(0) = 0$ . For (iii) see [18, Lemma A.6].  $\square$

*Proof of Theorem 4.1.* By Theorem 1.2  $F_{r,t} \rightarrow F$  locally uniformly in  $\mathbb{P}_0$ -probability as first  $t \uparrow \infty$  and then  $r \uparrow \infty$ . In particular, using Lemma 4.3 (i) we have that  $F_{r,t} \rightarrow F$  in  $M_1$ -topology in  $\mathbb{P}_0$ -distribution, that is for all bounded  $\varphi$  acting on  $D([0, \infty), \mathbb{R}_+)$  which are continuous in  $M_1$ -topology on a set with full  $\mathbb{P}_0$ -measure,

$$\lim_{r \uparrow \infty} \lim_{t \uparrow \infty} \mathbb{E}[E_0[\varphi(F_{r,t})]] = \mathbb{E}[E_0[\varphi(F)]]. \quad (4.15)$$

Now, observe that for any bounded continuous  $f$  on  $L^1_{\text{loc}}$ ,

$$\mathbb{E}[E_0[f(\mathcal{B}_{r,t}) - f(\mathcal{B})]] = \mathbb{E}[E_0[f \circ \pi(F_{r,t}, B) - f \circ \pi(F, B)]], \quad (4.16)$$

where

$$\pi : (D_{\uparrow}([0, \infty), \mathbb{R}_+), d_{M_1}) \times (D([0, \infty), \mathbb{R}_+), d_{J_1}) \rightarrow L_{\text{loc}}^1 \quad (a, w) \mapsto w \circ a^{-1}.$$

Note that the origin is contained in  $\mathcal{X}$  so that  $F^{-1}(0) = 0$  under  $P_0$ . Thus Lemma 4.3 (ii) and (iii) ensure the continuity of the mapping  $\pi$  in  $M_1$ -topology on a set with full  $\mathbb{P}_0$ -measure. Hence, (4.13) follows from (4.15).  $\square$

*Remark 4.4.* For  $x > 0$  a similar weak convergence result as in Theorem 4.1 holds under the annealed measure  $\int P_x() d\mathbb{P}$ . However, one needs to exclude time  $s = 0$  and obtains weak convergence in  $D((0, \infty), \mathbb{R}_+)$  only. This is because  $x$  might not be contained in the support  $\mathcal{X}$  of the random measure  $M$ , in which case  $F^{-1}(0) = 0$  does not hold.

## 5. THE SUBCRITICAL CASE

Recall that the McKean-martingale is defined as

$$Y_t^\sigma := \sum_{i=1}^{n(t)} e^{\sqrt{2}\sigma x_k(t) - (1+\sigma^2)t}, \quad \sigma \in (0, 1), \quad (5.1)$$

which is normalised to have mean 1. By [13, Theorem 4.2] the limit

$$Y^\sigma := \lim_{t \uparrow \infty} Y_t^\sigma \quad (5.2)$$

exists  $\mathbb{P}$ -a.s. and in  $L^1(\mathbb{P})$ . For  $v, r \in \mathbb{R}_+$  and  $t > r$ , we define a truncated version of the McKean-martingale  $Y_t^\sigma$  by

$$Y^\sigma(v, r, t) := \sum_{j \leq n(t)} e^{\sqrt{2}\sigma x_j(t) - (1+\sigma^2)t} \mathbb{1}_{\{\gamma(x_i(r)) \leq v\}}. \quad (5.3)$$

**Proposition 5.1.** *For each  $v \in \mathbb{R}_+$  the limit*

$$Y^\sigma(v) := \lim_{r \uparrow \infty} \lim_{t \uparrow \infty} Y^\sigma(v, r, t) \quad (5.4)$$

*exists  $\mathbb{P}$ -a.s. In particular,  $0 \leq Y^\sigma(v) \leq Y^\sigma$ . Moreover,  $Y^\sigma(v)$  is increasing in  $v$  and the corresponding Borel measure  $M^\sigma$  on  $\mathbb{R}_+$ , defined via  $M^\sigma([0, v]) = Y^\sigma(v)$  for all  $v \in \mathbb{R}_+$ , is  $\mathbb{P}$ -a.s. non-atomic.*

*Proof.* This follows by the same arguments as in [12, Proposition 3.2]. Observe that  $Y^\sigma(v, r, t)$  is non-negative by definition.  $\square$

Our goal is to state an analogue to Theorem 1.2 for the subcritical case. This will be done in Subsection 5.2 below. One of the main ingredients in the proof of Theorem 1.2 was the observation that the limiting measure  $M$  appears in the description of the extended extremal process for BBM as proved in [12, Theorem 3.1]. To proceed analogously in the subcritical case we need to notice that the martingales  $Y^\sigma$  with  $\sigma < 1$  appear in the description of the limiting extremal process of two speed branching Brownian motion and that the extended convergence result can be transferred to this class of models. This is the purpose of Subsection 5.1.

**5.1. The extremal process of two-speed branching Brownian motion.** Next we recall the characterisation of the extremal process for a two-speed branching Brownian motion established in [13]. For a fixed time  $u$ , a two-speed BBM is defined similarly as the ordinary BBM but at time  $t'$  the particles move as independent Brownian motions with variance

$$\sigma^2(t') = \begin{cases} \sigma_1^2, & 0 \leq t' < bu, \\ \sigma_2^2, & bu \leq t' \leq u, \end{cases} \quad 0 < b \leq 1, \quad (5.5)$$

where the total variance is normalised by assuming  $b\sigma_1^2 + (1-b)\sigma_2^2 = 1$ . Then, if  $\sigma_1 < \sigma_2$  the limit  $Y^{\sigma_1}$  of the McKean-martingale appears in the extremal process of the two-speed BBM. More precisely, we have the following result proven in [13, Theorem 1.2].

**Theorem 5.2.** *Let  $\tilde{x}_k(u)$  be a branching Brownian motion with variable speed  $\sigma^2(t')$  as given in (5.5). Assume that  $\sigma_1 < \sigma_2$ . Then,*

- (i)  $\lim_{u \uparrow \infty} \mathbb{P}(\max_{k \leq n(u)} \tilde{x}_k(u) - \tilde{m}(u) \leq y) = \mathbb{E}[\exp(-C(\sigma_2)Y^{\sigma_1}e^{-\sqrt{2}y})]$ ,  
where  $\tilde{m}(u) = \sqrt{2}u - \frac{1}{2\sqrt{2}} \log u$  and  $C(\sigma_2)$  is a constant depending on  $\sigma_2$ .
- (ii) *The point process*

$$\sum_{k \leq n(u)} \delta_{\tilde{x}_k(u) - \tilde{m}(u)} \Rightarrow \sum_{i,j} \delta_{\eta_i + \sigma_2 \Lambda_j^{(i)}} \quad \text{as } u \uparrow \infty \text{ in law.} \quad (5.6)$$

Here  $\eta_i$  denotes the  $i$ -th atom of a mixture of Poisson point process with intensity measure  $C(\sigma_2)Y^{\sigma_1}e^{-\sqrt{2}y}dy$  with  $C(\sigma_2)$  as in (i), and  $\Lambda_j^{(i)}$  are the atoms of independent and identically distributed point processes  $\Lambda_j^{(i)}$ , which are the limits in law of

$$\sum_{k \leq n(u)} \delta_{\bar{x}_k(u) - \max_{j \leq n(u)} \bar{x}_j(u)}, \quad (5.7)$$

where  $\bar{x}(u)$  is a BBM of speed 1 conditioned on  $\max_{j \leq n(u)} \bar{x}_j(u) \geq \sqrt{2}\sigma_2 t$ .

Using the embedding  $\gamma$  the convergence result in Theorem 5.2 can be extended as follows.

**Theorem 5.3.** *The point process*

$$\sum_{k=1}^{n(t)} \delta_{(\gamma(u^k(u)), \tilde{x}_k(u) - \tilde{m}(u))} \Rightarrow \sum_{i,j} \delta_{(q_i, p_i) + (0, \Lambda_j^{(i)})} \quad (5.8)$$

in law on  $\mathbb{R}_+ \times \mathbb{R}$ , as  $u \uparrow \infty$ , where  $(q_i, p_i)_{i \in \mathbb{N}}$  are the atoms of a Cox process on  $\mathbb{R}_+ \times \mathbb{R}$  with intensity measure  $M^{\sigma_1}(dv) \times C(\sigma_2)e^{-\sqrt{2}x}dx$ , where  $M^{\sigma_1}(dv)$  is the random measure on  $\mathbb{R}_+$  characterised in Proposition 5.1, and  $\Lambda_j^{(i)}$  are the atoms of independent and identically distributed point processes  $\Lambda^{(i)}$  as in Theorem 5.2 (ii).

*Proof.* The proof goes along the lines of the proof of [12, Theorem 3.1]. Note that by the localisation of the path of extremal particles given in [13, Proposition 2.1]

the thinning can be applied in the same way using [13, Proposition 3.1] which provides the right tail bound on the maximum. This gives an alternative way to get the convergence of the local maxima to a Poisson point process. There the McKean-martingale  $Y_t^{\sigma_1}$  appears naturally instead of the derivative martingale and one proceeds as in the proof of [12, Theorem 3.1].  $\square$

**5.2. Approximation of the PCAF and the process.** Similarly as in the critical case, for any fixed  $\sigma \in (0, 1)$  we define the measure  $M_{r,t}^\sigma$  on  $\mathbb{R}_+$  associated with  $Y^\sigma(\cdot, r, t)$  by

$$M_{r,t}^\sigma := \sum_{j \leq n(t)} e^{\sqrt{2}\sigma x_j(t) - (1+\sigma^2)t} \delta_{\gamma(x_j(r))}. \quad (5.9)$$

Then Theorem 5.3 implies that  $\mathbb{P}$ -a.s.

$$M^\sigma = \lim_{r \uparrow \infty} \lim_{t \uparrow \infty} M_{r,t}^\sigma \quad \text{weakly.} \quad (5.10)$$

Again we are aiming to lift this convergence on the level of the associated PCAFs.

**Proposition 5.4.** *Let  $\sigma \in (0, 1)$  be fixed. Then,  $\mathbb{P}$ -a.s., for any  $0 \leq r < t$  the following hold.*

(i) *The unique PCAF of  $B$  with Revuz measure  $M_{r,t}^\sigma$  is given by*

$$F_{r,t}^\sigma : [0, \infty) \rightarrow [0, \infty) \quad s \mapsto \sum_{j=1}^{n(t)} e^{\sqrt{2}\sigma x_j(t) - (1+\sigma^2)t} L_s^{\gamma(x_j(r))}. \quad (5.11)$$

(ii) *For all  $x \in \mathbb{R}_+$ ,  $P_x$ -a.s.,  $F_{r,t}^\sigma$  is continuous, increasing and satisfies  $F_{r,t}^\sigma(0) = 0$  and  $\lim_{s \rightarrow \infty} F_{r,t}^\sigma(s) = \infty$ .*

*Proof.* This is again a direct consequence from the properties of Brownian local times in Lemma B.3 and B.1. Note that in this setting the positivity is clear since  $\exp$  is a positive function.  $\square$

**Theorem 5.5.** *Let  $\sigma \in (0, 1)$  be fixed. For any  $S > 0$  and  $x \in \mathbb{R}_+$  the following hold.*

- (i) *The limit  $F^\sigma = \lim_{r \uparrow \infty} \lim_{t \uparrow \infty} F_{r,t}^\sigma$  exists in sup-norm on  $[0, S]$  in  $\mathbb{P} \times P_x$ -probability.*
- (ii) *The limiting functional  $F^\sigma$  is  $\mathbb{P}_x$ -a.s. explicitly given by*

$$F^\sigma(s) = \int_{\mathbb{R}_+} L_s^a M^\sigma(da), \quad s \geq 0, \quad (5.12)$$

*and  $F^\sigma$  is  $\mathbb{P}$ -a.s. the (up to equivalence) unique PCAF with Revuz measure  $M^\sigma$ . Furthermore,  $F^\sigma$  is continuous, increasing and satisfies  $F^\sigma(0) = 0$  and  $\lim_{s \rightarrow \infty} F^\sigma(s) = \infty$ .*

*Proof.* This follows by similar arguments as in the proof of Theorem 1.2 above. Note that the proof of the  $\mathbb{P}$ -a.s. limit of  $F_{r,t}$  in  $t$  is simpler because only independent copies of  $Y^\sigma$  appear in the analogue of (3.4).  $\square$

Now we define the process  $\mathcal{B}^\sigma(s) := B_{(F^\sigma)^{-1}(s)}$ ,  $s \geq 0$ . Similarly as explained in Section 4 above for  $\mathcal{B}$ , by the general theory of time changes of Markov processes the process  $\mathcal{B}^\sigma$  is a recurrent,  $M^\sigma$ -symmetric pure jump diffusion on the support of  $M^\sigma$  and its Dirichlet form can be abstractly described. For  $0 < r < t$  let

$$\mathcal{B}_{r,t}^\sigma(s) := B_{F_{r,t}^{-1}(s)}^\sigma, \quad s \geq 0. \quad (5.13)$$

Then, from Theorem 5.5 we obtain as in the critical case the convergence of the associated process.

**Theorem 5.6.** *Under the annealed law  $\int P_0() d\mathbb{P}$ ,*

$$\lim_{r \uparrow \infty} \lim_{t \uparrow \infty} \mathcal{B}_{r,t}^\sigma = \mathcal{B}^\sigma \quad (5.14)$$

*in distribution on  $L_{\text{loc}}^1$ .*

*Proof.* This can be shown by the same arguments as Theorem 4.1.  $\square$

#### APPENDIX A. ADDITIVE FUNCTIONALS

In this section we briefly recall the definition of an additive functional of a symmetric Markov process and some of its main properties, for more details on this topic see e.g. [24, 16]. Let  $E$  be a locally compact separable metric space and let  $m$  be a positive Radon measure on  $E$  with  $\text{supp}(m) = E$ . We consider an  $m$ -symmetric conservative Markov process  $(\Omega', \mathcal{G}, (\mathcal{G}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (P_x)_{x \in E})$  and denote by  $\{\theta_t\}_{t \geq 0}$  the family of shift mappings on  $\Omega'$ , i.e.  $X_{t+s} = X_t \circ \theta_s$  for  $s, t \geq 0$ .

**Definition A.1.** i) A  $[0, \infty]$ -valued stochastic process  $A = (A_t)_{t \geq 0}$  on  $(\Omega', \mathcal{G})$  is called a *positive continuous additive functional (PCAF)* of  $X$  (in the strict sense), if  $A_t$  is  $\mathcal{G}_t$ -measurable for every  $t \geq 0$  and if there exists a set  $\Lambda \in \mathcal{G}$ , called a *defining set* for  $A$ , such that

- a) for all  $x \in E$ ,  $P_x[\Lambda] = 1$ ,
- b) for all  $t \geq 0$ ,  $\theta_t(\Lambda) \subset \Lambda$ ,
- c) for all  $\omega \in \Lambda$ ,  $[0, \infty) \ni t \mapsto A_t(\omega)$  is a  $[0, \infty)$ -valued continuous function with  $A_0(\omega) = 0$  and

$$A_{t+s}(\omega) = A_t(\omega) + A_s \circ \theta_t(\omega), \quad \forall s, t \geq 0. \quad (\text{A.1})$$

ii) Two such functionals  $A^1$  and  $A^2$  are called *equivalent* if  $P_x[A_t^1 = A_t^2] = 1$  for all  $t > 0$ ,  $x \in E$ , or equivalently, there exists a defining set  $\Lambda \in \mathcal{G}_\infty$  for both  $A^1$  and  $A^2$  such that  $A_t^1(\omega) = A_t^2(\omega)$  for all  $t \geq 0$ ,  $\omega \in \Lambda$ .

iii) For any such  $A$ , a Borel measure  $\mu_A$  on  $E$  satisfying

$$\int_E f(y) \mu_A(dy) = \lim_{t \downarrow 0} \frac{1}{t} \int_E E_x \left[ \int_0^t f(B_s) dA_s \right] m(dx) \quad (\text{A.2})$$

for any non-negative Borel function  $f : E \rightarrow [0, \infty]$  is called the *Revuz measure* of  $A$ , which exists uniquely by general theory (see e.g. [16, Theorem A.3.5]).

We recall that for a given a Borel measure  $\mu_A$  charging no polar sets a PCAF  $A$  satisfying (A.2) exists uniquely up to equivalence (see e.g. [24, Theorem 5.1.3]). Observe that in the present setting where  $m$  is invariant the measure  $\mu_A$  is already characterised by the simpler formula

$$\int_E f(y) \mu_A(dy) = \int_E E_x \left[ \int_0^1 f(B_s) dA_s \right] m(dx). \quad (\text{A.3})$$

## APPENDIX B. BROWNIAN LOCAL TIMES

In this section we consider Brownian local times as an example for a PCAF on the Wiener space and recall some of their properties needed in the present paper. Let  $(\Omega', \mathcal{G}, (\mathcal{G}_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$  be the Wiener space as introduced in Section 2 with coordinate process  $W$ , so that  $B := |W|$  becomes a reflected Brownian motion on  $\mathbb{R}_+$  with a field of local times denoted by  $\{L_t^a, t \geq 0, a \in \mathbb{R}_+\}$ .

**Lemma B.1.** *There exists a set  $\Lambda \subset \Omega'$  with  $P_x[\Lambda] = 1$  for all  $x \in \mathbb{R}_+$  such that for all  $\omega' \in \Lambda$  the following hold.*

- (i) *For every  $a \in \mathbb{R}_+$  the mapping  $t \mapsto L_t^a$  is continuous, increasing and satisfies  $L_0^a(\omega') = 0$  and  $\lim_{t \rightarrow \infty} L_t^a(\omega') = \infty$ . The measure  $dL_t^a(\omega')$  is carried by the set  $\{t \geq 0 : B_t(\omega) = a\}$ .*
- (ii) *The mapping  $(a, t) \mapsto L_t^a(\omega')$  is jointly continuous and for every  $\alpha < 1/2$  and  $T > 0$  there exists  $C_1 = C_1(\omega', \alpha, T)$  satisfying  $\sup_{x \in \mathbb{R}_+} E_x[C_1] < \infty$  such that*

$$\sup_{t \leq T} |L_t^a(\omega') - L_t^b(\omega')| \leq C_1 |a - b|^\alpha. \quad (\text{B.1})$$

*Proof.* These properties are immediate from (2.9) since the Brownian local time  $L(W)$  satisfies them. We refer to [31, Chapter VI] for details, in particular [31, Corollary VI.2.4] for (i) and [31, Theorem VI.1.7 and Corollary VI.1.8] for (ii) (cf. also [24, Example 5.1.1]).  $\square$

**Lemma B.2.** *For any  $t > 0$  there exists  $\lambda_0 = \lambda_0(t) > 0$  and a positive constant  $C_2$  such that*

$$P_x \left[ \sup_{a \in \mathbb{R}_+} L_t^a > \lambda \right] \leq C_2 \frac{\lambda}{\sqrt{t}} e^{-\lambda^2/2t}, \quad \forall x \in \mathbb{R}_+, \lambda \geq \lambda_0. \quad (\text{B.2})$$

*In particular,  $\sup_{a \in \mathbb{R}_+} L_t^a \in L^2(P_x)$  for any  $x \in \mathbb{R}_+$ .*

*Proof.* In view of (2.9) it suffices to consider the local times  $L_t^a(W)$  of the standard Brownian motion  $W$ . Note that the event  $\{\sup_{a \in \mathbb{R}} L_t^a(W) > \lambda\}$  does not depend on the starting point of  $W$ . Under  $P_0$  the tail estimate in (B.2) for  $\sup_{a \in \mathbb{R}} L_t^a(W)$  has been shown in [19, Lemma 1]. The fact that  $\sup_{a \in \mathbb{R}_+} L_t^a \in L^2(P_x)$  follows from (B.2) by integration.  $\square$

Recall that in dimension one only the empty set is polar for  $W$  or  $B$ , so trivially any  $\sigma$ -finite measure  $\mu$  on  $\mathbb{R}$  does not charge polar sets and by general theory (see



e.g. [16, Theorem 4.1.1]) there exist unique (up to equivalence) PCAF  $A$  of  $W$  or  $B$  with  $\mu_A = \mu$ . In particular, for any  $a \in \mathbb{R}$  the unique PCAF of  $W$  having the Dirac measure  $\delta_a$  as Revuz measure is given by  $L^a(W)$ , see [24, Example 5.1.1] or [31, Proposition X.2.4]. This can be easily transferred to the reflected Brownian motion.

**Lemma B.3.** *For any  $a \in \mathbb{R}_+$  the local time  $L^a$  is the PCAF of  $B$  with Revuz measure  $\delta_a$ .*

*Proof.* We need to show that for any non-negative Borel function  $f$  on  $\mathbb{R}_+$ ,

$$f(a) = \lim_{t \downarrow 0} \frac{1}{t} \int_{\mathbb{R}_+} E_x \left[ \int_0^t f(B_s) dL_s^a \right] dx. \quad (\text{B.3})$$

We extend  $f$  to a function  $\tilde{f}$  on  $\mathbb{R}$  by setting  $\tilde{f}(x) := f(|x|)$ ,  $x \in \mathbb{R}$ . Using that  $L^a(W)$  is the unique PCAF of  $W$  with  $\mu_{L^a(W)} = \delta_a$  and that for any  $x \in \mathbb{R}$  the measure  $dL^a(W)$  is  $P_x$ - a.s. carried by the set  $\{t : W_t = a\}$  we have

$$\begin{aligned} f(a) &= \tilde{f}(a) = \lim_{t \downarrow 0} \frac{1}{t} \int_{\mathbb{R}} E_x \left[ \int_0^t \tilde{f}(W_s) dL_s^a(W) \right] dx \\ &= \lim_{t \downarrow 0} \frac{1}{t} \int_{\mathbb{R}_+} E_x \left[ \int_0^t f(B_s) dL_s^a(W) \right] dx + \lim_{t \downarrow 0} \frac{1}{t} \int_{-\infty}^0 E_x \left[ \int_0^t \tilde{f}(W_s) dL_s^a(W) \right] dx. \end{aligned} \quad (\text{B.4})$$

Since  $L^a(-W) = L^{-a}(W)$  (cf. [31, Exercise VI.1.17]) we get

$$\begin{aligned} \int_{-\infty}^0 E_x \left[ \int_0^t \tilde{f}(W_s) dL_s^a(W) \right] dx &= \int_{-\infty}^0 E_{-x} \left[ \int_0^t \tilde{f}(-W_s) dL_s^a(-W) \right] dx \\ &= \int_{\mathbb{R}_+} E_x \left[ \int_0^t f(B_s) dL_s^{-a}(W) \right] dx \end{aligned} \quad (\text{B.5})$$

and combining this with (B.4) and (2.9) we obtain (B.3).  $\square$

**Lemma B.4.** *For any  $a \in \mathbb{R}_+$ ,*

$$\int_{\mathbb{R}_+} E_x[L_1^a] dx = 1. \quad (\text{B.6})$$

*Proof.* Recall that

$$E_x[L_1^a] = E_x[L_1^a(W)] + E_x[L_1^{-a}(W)] = E_a[L_1^x(W)] + E_a[L_1^{-x}(W)]. \quad (\text{B.7})$$

Hence, by the occupation times formula we obtain

$$\int_{\mathbb{R}_+} E_x[L_1^a] dx = E_a \left[ \int_{-\infty}^{\infty} L_1^x(W) dx \right] = 1 \quad (\text{B.8})$$

(cf. [31, proof of Proposition X.2.4]).  $\square$

*Acknowledgment.* We thank Anton Bovier, Zhen-Qing Chen and Takashi Kumagai for useful discussions and valuable comments. We are grateful to an anonymous referee for pointing out an error in an earlier version of Theorem 4.1.

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